A NOTE ON INTERSECTION THEORY OF DIVISORS (LINE BUNDLES)

All varieties in this note are projective variety over $\mathbb{C}$, except otherwise stated.

Definition 0.1. Let X be a variety and $\mathcal{F}$ be a coherent sheaf on X. Suppose that the dimension of support of $\mathcal{F}$ is at most n, and take n invertible sheaves $L_1, \cdots, L_n$ on X. Define the intersection of $L_1, \cdots, L_n$ with $\mathcal{F}$ by

$$(L_1 \cdots L_n \cdot \mathcal{F}) = \sum_{\{i_1,\cdots,i_m\} \subseteq \{1,\cdots,n\}} (-1)^m \chi(X, L_{i_1}^\vee \otimes \cdots \otimes L_{i_m}^\vee \otimes \mathcal{F}). \tag{0.2}$$

If Y is a subvariety of X of dimension at most n, we denote $(L_1 \cdots L_n \cdot \mathcal{O}_Y)$ by $(L_1 \cdots L_n \cdot Y)$, and when $(L_1 \cdots L_n \cdot X)$ by $(L_1 \cdots L_n)$. If $D_1, \cdots, D_n$ are Cartier divisors, we write

$$(D_1 \cdots D_n \cdot \mathcal{F}) := (\mathcal{O}(D_1) \cdots \mathcal{O}(D_n) \cdot \mathcal{F}). \tag{0.3}$$

The following are the main useful properties of this intersection product:

Proposition 0.4. *The intersection product defined above satisfies*

1. *If $L_1 = \mathcal{O}_X$, then $(L_1 \cdots L_n \cdot \mathcal{F}) = 0$.*
2. *$(L_1 \cdots L_n \cdot \mathcal{F})$ is symmetric and multilinear in L_i.*
3. *For subscheme Z of X of dimension n, we have $(L_1 \cdots L_n \cdot Z) = (L_1|_Z \cdots L_n|_Z)$.*
4. *(Projection formula) Let $f : Y \to X$ be a morphism of varieties, and $\mathcal{G}$ be a coherent sheaf on Y*

$$(f^*L_1 \cdots f^*L_n \cdot \mathcal{G}) = (L_1 \cdots L_n \cdot f_*\mathcal{G}) \tag{0.5}$$

5. *If $f : Y \to X$ is a morphism of varieties of the same dimension n, then*

$$(f^*L_1 \cdots f^*L_n) = \deg(f)(L_1 \cdots L_n) \tag{0.6}$$

We leave proving (1) as an exercise and start by proving (2). Symmetry is immediate from definition. For multilinearity, we need following lemma

Lemma 0.7. *Suppose that $L_n = \mathcal{O}_X(D)$ such that D is effective cartier divisor which does not contain any associated point of $\mathcal{F}$, then*

$$(L_1 \cdots L_n \cdot \mathcal{F}) = (L_1|_D \cdots L_{n-1}|_D \cdot \mathcal{F}|_D). \tag{0.8}$$

Proof. We have the usual exact sequence

$$0 \to \mathcal{O}(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0 \tag{0.9}$$

Locally, the first map is given by multiplication by the equation defining D. It follows that by tensoring with $\mathcal{F}$, we gen injective map $\mathcal{F}(-D) \to \mathcal{F}$ because if D is locally defined on affine open $\mathrm{Spec}(A)$ by a, then a cannot be zero divisor of the module corresponding to $\mathcal{F}$ as otherwise a will vanish on an associated point $\mathcal{F}$. At the end, we get exact sequence

$$0 \to \mathcal{O}(-D) \to \mathcal{F} \to \mathcal{F}|_D \to 0 \tag{0.10}$$

which implies by additivity of Euler characteristic with respect to exact sequences

(0.11)

$$
\begin{aligned}
(L_1|_D \cdots L_{n-1}|_D \cdot \mathcal{F}|_D) &= \sum_{\{j_1,\cdots,j_{m-1}\}\subseteq\{1,\cdots,n-1\}} (-1)^{m-1}\chi(D, L_{i_1}^\vee|_D \otimes \cdots \otimes L_{i_{m-1}}^\vee|_D \otimes \mathcal{F}|_D) \\
&= \sum_{\{j_1,\cdots,j_{m-1}\}\subseteq\{1,\cdots,n-1\}} (-1)^{m-1}\chi(D, L_{i_1}^\vee \otimes \cdots \otimes L_{i_{m-1}}^\vee \otimes i_*\mathcal{F}|_D) \\
&= -\sum_{\{i_1,\cdots,i_{m-1},n\}\subseteq\{1,\cdots,n\}} (-1)^{m-1}\chi(X, L_{i_1}^\vee \otimes \cdots \otimes L_{i_{m-1}}^\vee \otimes \mathcal{F}(-D))+ \\
&\quad + \sum_{\{i_1,\cdots,i_m\}\subseteq\{1,\cdots,n-1\}} (-1)^{m}\chi(X, L_{i_1}^\vee \otimes \cdots \otimes L_{i_{m-1}}^\vee \otimes \mathcal{F}) \\
&= \sum_{\{i_1,\cdots,i_{m-1},n\}\subseteq\{1,\cdots,n\}} (-1)^{m}\chi(X, L_{i_1}^\vee \otimes \cdots \otimes L_{i_{m-1}\otimes L_n^\vee}^\vee \otimes \mathcal{F}) \\
&\quad + \sum_{\{i_1,\cdots,i_m\}\subseteq\{1,\cdots,n-1\}} (-1)^{m}\chi(X, L_{i_1}^\vee \otimes \cdots \otimes L_{i_{m-1}}^\vee \otimes \mathcal{F}) \\
&= \sum_{\{i_1,\cdots,i_m\}\subseteq\{1,\cdots,n\}} (-1)^{m}\chi(X, L_{i_1}^\vee \otimes \cdots \otimes L_{i_m}^\vee \otimes \mathcal{F}) \\
&= (L_1 \cdots L_n \cdot \mathcal{F})
\end{aligned}
$$

In the second equality we uese the projection formula $\square$

Now, we prove the original claim by induction on n. The base case $n = 0$ is trivial. Next, let's observe that

(0.12)

$$(L_1 \cdot L_1' \cdot L_2 \cdots L_n \cdot \mathcal{F}) = (L_1 \cdot L_2 \cdots L_n \cdot \mathcal{F}) + (L_1' \cdot L_2 \cdots L_n \mathcal{F}) - (L_1 \otimes L_1' \cdot L_2 \cdots L_n \cdot \mathcal{F}).$$

It suffices to prove that the left hand side is equal to 0 by symmetry. Let D be an effective Cartier divisor avoiding the associated points of $\mathcal{F}$. We know that $\mathcal{F}|_D$ has support of dimension at most $n-1$ by Krull principal ideal theorem, then by inductive hypothesis letting $L_n = \mathcal{O}(D)$, we have

$$(L_1 \cdot L_1' \cdots L_n \cdot \mathcal{F}) = (L_1 \cdot L_1' \cdots L_{n-1} \cdot \mathcal{F}|_D) = 0. \tag{0.13}$$

For ample line bundle L, we can find an effective cartier divisor D avoiding the associated points of $\mathcal{F}$ and $L = \mathcal{O}(D)$ (see the proof of (5) for more details). We conclude that for any ample line bundle L, we have

$$(L \cdot L' \cdot L_2 \ldots L_n \cdot \mathcal{F}) = 0 \tag{0.14}$$

That means (by symmetry) that if L is ample line bundle, then

$$(L \cdot L_2 \cdots L_n \cdot \mathcal{F}) \tag{0.15}$$

is linear in L_n. It is well-known that a line bundle L_1 can be written as $L_1 = A \otimes B^\vee$, where A and B are ample line bundles. Putting $L = B$ and $L' = A \otimes B^\vee$ in (0.14) and using (0.12), we get

$$(L \otimes K^\vee \cdot L_2 \cdots L_n \cdot \mathcal{F}) = (L \cdot L_2 \cdots L_n \cdot \mathcal{F}) - (K \cdot L_2 \cdots L_n \cdot \mathcal{F}). \tag{0.16}$$

So, the right side is multilinear in L_n by (0.15). We are done by symmetry.

By the proof above, we have the following lemma, which will be used

Lemma 0.17. *If $n > \dim \operatorname{Supp}(\mathcal{F})$, then*

$$(L_1 \cdots L_n \cdot \mathcal{F}) = 0 \tag{0.18}$$

The projection formula (4) is proven as follows. We recall Grothendieck spectral sequence

$$E_2^{p,q} = H^p(R^q f_*(f^*L_{i_1}^\vee \otimes \cdots \otimes f^*L_{i_m}^\vee \otimes \mathcal{G})) \implies H^{p+q}(f^*L_1^\vee \otimes \cdots \otimes f^*L_n^\vee \otimes \mathcal{G}) \tag{0.19}$$

However, by the projection formula of sheaves, the E_2 term is canonically isomorphic to

$$E_2^{p,q} = H^p(L_{i_1}^\vee \otimes \cdots \otimes L_{i_m}^\vee \otimes R^q f_*\mathcal{G}) \tag{0.20}$$

It follows that (as the Euler characteristic of the pages of a spectral sequence are equal)

$$\sum_{p,q} (-1)^{p+q} h^p(L_{i_1}^\vee \otimes \cdots \otimes L_{i_m}^\vee \otimes R^q f_*\mathcal{G}) = \sum_i (-1)^i h^i(f^*L_{i_1}^\vee \otimes \cdots \otimes f^*L_{i_m}^\vee \otimes \mathcal{G}) \tag{0.21}$$

Hence,

(0.22)

$$\begin{aligned}
(f^*L_1 \cdots f^*L_n \cdot \mathcal{G}) &= \sum_{\{i_1,\cdots,i_m\} \subseteq \{1,\cdots,n\}} (-1)^m \chi(f^*L_{i_1}^\vee \otimes f^*Li_m{}^\vee \otimes \mathcal{G}) \\
&= \sum_{\{i_1,\cdots,i_m\} \subseteq \{1,\cdots,n\}} (-1)^m \sum_{p,q} (-1)^{p+q} h^p(L_{i_1}^\vee \otimes \cdots \otimes L_{i_m}^\vee \otimes R^q f_*\mathcal{G}) \\
&= \sum_{\{i_1,\cdots,i_m\} \subseteq \{1,\cdots,n\}} (-1)^q \sum (-1)^m \chi(L_{i_1}^\vee \otimes \cdots \otimes L_{i_m}^\vee \otimes R^q f_*\mathcal{G}) \\
&= \sum_q (-1)^q (L_1 \cdots L_n \cdot R^q f_*\mathcal{G})
\end{aligned}$$

It remains to show that $(L_1 \cdots L_n \cdot R^q f_*\mathcal{G}) = 0$ for $q > 0$. For this, it suffices by lemma 0.17 to show that $\operatorname{Supp}(R^q f_*\mathcal{G})$ has dimension less than n. It is clear that the support of $R^q f_*\mathcal{G}$ lies inside the image of the support of $\mathcal{G}$ (Recall that $R^q f_*\mathcal{G}$ is the sheafification of $U \mapsto H^q(f^{-1}(U), \mathcal{G})$.) It suffices then to show that for each irreducible component of Z of $\operatorname{Supp}(\mathcal{G})$ such that $f(Z)$ has dimension n, we have $(R^q f_*\mathcal{G})_x = 0$ for generic x in $f(Z)$. It is not so difficult (if we assume we know openess of flat locus) to find open $U \subseteq f(Z)$ such that the restriction of f to $V = f^{-1}(U) \cap Z$ is finite and $\mathcal{G}$ is flat over U. Now we apply cohomology and Base change theorems. By finitenes of $f|_V$, we know by Grothendieck vanishing theorem that

$$\dim H^q(f^{-1}(x), \mathcal{G}_x) = 0 \tag{0.23}$$

for all $q > 0$, we find by Grauret theorem that $R^q f_*\mathcal{G}$ is locally free sheaf on U of rank 0. It follows that the support avoids U, that's the support meets $f(Z)$ in a subvariety of dimension strictly less than n, which is what we wanted.

For (3), this follows immediately from projection formula by taking $f = i$, where $i : Z \to X$ is the closed immersion.

Finally, we prove (5). By projection formula,

$$(f^*L_1 \cdots f^*L_n) = (L_1 \cdots L_n \cdot f_*\mathcal{O}_X) \tag{0.24}$$

We know that there is a dense open subvariety U such that $(f_*\mathcal{O}_X)|_U$ is locally free of rank $\deg(f)$. Next, write $Z = X\backslash U$, and endow it with the reduced structure. By multilinearity, we can assume that $L_1, \cdots, L_n$ are very ample. Now, we claim there is a sequence of subschemes $D_1 \subset \cdots \subset D_n$ such that D_i is effective Cartier divisor in D_{i+1} such that $L_i|_{D_{i+1}} = \mathcal{O}_{D_{i+1}}(D_i)$ and $D_1 \cap Z = \emptyset$. Assuming this claim for now, we have inductively using lemma 0.7

$$\begin{aligned}(L_1 \cdots L_n \cdot f_*\mathcal{O}_X) &= (L_1|_{D_{i+1}} \cdots L_i|_{D_{i+1}} \cdot (f_*\mathcal{O}_X)|_{D_{i+1}}) \\ &= (L_1|_{D_{i+1}} \cdots L_i|_{D_i} \cdot \mathcal{O}|_{D_{i+1}}(D_i) \cdot (f_*\mathcal{O}_X)|_{D_{i+1}}) \\ &= (L_1|_{D_i} \cdots L_{i-1}|_{D_i} \cdot (f_*\mathcal{O}_X)|_{D_i}) \\ &= (\cdot(f_*\mathcal{O}_X)|_{D_1}) = (\cdot\mathcal{O}_{D_1}^{\deg(f)}) \\ &= \deg(f)(L_1|_{D_2} \cdot \mathcal{O}_{D_2}^{\deg(f)}) = \cdots = \deg(f)(L_1 \cdots L_n).\end{aligned} \tag{0.25}$$

Now we need to prove the claim above. Inductively, assume that $Z' = D_{i+1} \cap Z$ has dimension i, and embed D_{i+1} into $\mathbb{P}^N$ such that $L_i|_{D_{i+1}} = \mathcal{O}_{\mathbb{P}^N}(H)|_{D_{i+1}}$ where H is a hyperplane. It is known that for generic hyperplane H, the dimension of $H \cap Z'$ is one less the dimension of Z' (i.e equals $i-1$.) Moreover, one can always choose this hyperplane to avoid the finitely many associated points of D_i and hence $H \cap D_{i+1}$ is an effective Cartier divisor representing $L_i|_{D_{i+1}}$, we define $D_i = H \cap D_{i+1}$. The sequence $D_1 \subset \cdots D_n$ thus produced satisfy the claim.

We give a cohomological formula of the intersection formula, for this we start with the following definition of the cycle associated to a coherent sheaf.

Definition 0.26. If $\dim \operatorname{supp}(\mathcal{F}) \leq n$, then we define the class of $[F]$ by

$$[\mathcal{F}]_n = \sum_Y \operatorname{length}_{\mathcal{O}_{X,\xi_Y}}(\mathcal{F}_{\xi_Y})[Y^{an}] \in H_n^{BM}(X^{an}; \mathbb{Z}), \tag{0.27}$$

where the sum is over all irreducible components of $\operatorname{supp}(\mathcal{F})$ of dimensino n and ξ_Y denotes the generic point of Y. Here H_*^{BM} denotes the Borel-Moore cohomology and $[Y^{an}]$ the fundamental class of Y^{an}.

Proposition 0.28. *Let X be a variety, $\mathcal{F}$ be a coherent sheaf on X and $L_1, \cdots, L_n$ be line bundles on n. Suppose that the dimension of the support on $\mathcal{F}$ is less than or equal n. Then,*

$$(L_1 \cdots L_n \cdot \mathcal{F}) = (c_1(L_1^{an}) \cup \cdots \cup c_1(L_n^{an})) \cap [\mathcal{F}] \tag{0.29}$$

Proof. We will prove this only for the case $\mathcal{F} = \mathcal{O}_Y$, where Y is a subscheme of X. Now by multilinearity of both sides, we can assume that L_i is ample for each i. The proof then is divided into two steps.

Step 1. We reduce to proving that for effective ample Cartier divisor D such that all the associated points of Y are not in D, we have $c_1(\mathcal{O}_X(D)^{an}) \cap [Y^{an}] = [D|_Y^{an}]$, where $D|_Y$ is the restriction D to Y.

This is similar to our proof of assertion (5) of the previous proposition. We recall that one can find sequence of subschemes $D_1 \subset \cdots \subset D_n$ such that D_i contains no associated point of $D_{i+1} \cap Z$. Inductively, we have

$$
\begin{aligned}
(L_1 \cdots L_n \cdot \mathcal{F}) &= (L_1|_D \cdots L_{n-1}|_D \cdot \mathcal{O}_{D|_Y}) \\
&= i^* c_1(L_1^{an}) \cup \cdots \cup i^* c_1(L_{n-1}^{an}) \cap [D|_Y] \\
&= (c_1(L_1^{an}) \cup \cdots \cup c_1(L_{n-1}^{an})) \cap i_*([D|_Y]) \\
&= (c_1(L_1^{an}) \cup \cdots \cup c_1(L_{n-1}^{an})) \cap (c_1(L_n^{an}) \cap [Y^{an}]) \\
&= (c_1(L_1^{an}) \cup \cdots \cup c_1(L_{n-1}^{an}) \cup c_1(L_n^{an})) \cap [Y^{an}],
\end{aligned} \tag{0.30}
$$

where in the last equality we used the property of cap and cup products

$$(\alpha \cup \beta) \cap c = \alpha \cap (\beta \cap c). \tag{0.31}$$

Step 2. We prove the statement in step 2. In the following we drop the "an" superscript. First, it suffices to show the statement for $Y = X$. The second thing to note that by ampleness, we can assume that $D' = D \cap X^{reg}$ is smooth (Bertini theorem), thus we can assume X is quasi-projective and smooth and D a smooth divisor on it [1]. We let ∇ be a Chern connection of $\mathcal{O}_X(D)$ with respect to a so that

$$c_1(\mathcal{O}_X(D)) = \left[\frac{\sqrt{-1}}{2\pi} F_\nabla\right] = -\frac{\sqrt{-1}}{2\pi} \partial\overline{\partial} \log h \tag{0.32}$$

We have to show that for all compactly supported closed forms ω

$$\int_X c_1(\mathcal{O}_X(D)) \cup \omega = \int_D \omega. \tag{0.33}$$

Let s be the section defining D, and let U_ϵ be the neighborhood

$$\{x \in X | \|s(x)\|_h < \epsilon\}.$$

Then we compute

$$
\begin{aligned}
\int_X c_1(\mathcal{O}_X(D)) \cup \omega &= \lim_{\epsilon \to 0} \frac{\sqrt{-1}}{2\pi} \int_{X \setminus D_\epsilon} F_\nabla \wedge \omega \\
&= \lim_{\epsilon \to 0} -\frac{\sqrt{-1}}{2\pi} \int_{X \setminus D_\epsilon} \partial\overline{\partial} \log \|s\|_h \wedge \omega \\
&= \lim_{\epsilon \to 0} \frac{\sqrt{-1}}{4\pi} \int_{\partial D_\epsilon} (\partial - \overline{\partial}) \log \|s\|_h \wedge \omega
\end{aligned} \tag{0.34}
$$

We can see that the problem is local, we take coordinate chart $(U, z_1, \cdots, z_n)$ on which L is trivial with s being given by z_1 in the trivialization, and the metric is given by h_i. We assume the image of the chart is the polydisc $\{|z_i| < 1\}$. We get

$$
\begin{aligned}
(\partial - \overline{\partial}) \log \|s\|_h &= \partial \log z_1 - \overline{\partial \log z_1} + (\partial - \overline{\partial}) \log h_i \\
&= 2\sqrt{-1} \mathrm{Im}\,(\partial \log z_1) + (\partial - \overline{\partial}) \log h_1
\end{aligned} \tag{0.35}
$$

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[1] Note the we used the following observation. There is a map

$$\mathrm{Div}(X) \to Z_{n-1}(X) \to H^{BM}_{2n-2}(X)$$

the first map was mentioned in Ruadhaí's talk and the second map is the obvious one. Now that composition sends our effective Cartier divisor D to the class $[D]_n$ of definition 0.26 and the important thing here is that the map factors through Cartier divisor equivalence classes and hence we can use any Cartier divisor linearly equivalent to D which we have just done using Bertini theorem.

The second summand does not contribute to the integral as $\epsilon \to 0$ because it is continuous. What we want to show then is

$$\lim_{\epsilon\to 0}\frac{1}{2\pi}\int_{\partial D_\epsilon\cap U}\mathrm{Im}(\partial\log z_1)\wedge\omega = -\int_{D\cap U}\omega \tag{0.36}$$

We have clearly

$$\partial\log(z_1) = \frac{dz_1}{z_1} \tag{0.37}$$

hence, we can assume that

$$\omega = f(dz_2\wedge\cdots\wedge dz_n)\wedge(d\overline{z}_2\wedge\cdots\wedge d\overline{z}_n) \tag{0.38}$$

We compute the right side of (0.36)

(0.39)
$$\int_{D\cap U}\omega = \int_{z_1=0}\omega = \int_{|z_2|<1,\cdots,|z_n|<1} f(0,z_2,\cdots,z_n)(dz_2\wedge\cdots\wedge dz_n)\wedge(d\overline{z}_2\wedge\cdots\wedge d\overline{z}_n)$$

and the left side

$$\int_{\partial D_\epsilon\cap U}\partial\log\|s\|_h\wedge\omega = -\int_{|h(z_1)|=\epsilon} f\frac{dz_1}{z_1}\wedge(dz_2\wedge\cdots\wedge dz_n)\wedge(d\overline{z}_2\wedge\cdots\wedge d\overline{z}_n) \tag{0.40}$$

By Cauchy integral formula, we can reduce this as follows

(0.41)
$$\begin{aligned}
&-\frac{1}{2\pi}\mathrm{Im}\int_{|h(z_1)|=\epsilon} f\frac{dz_1}{z_1}\wedge(dz_2\wedge\cdots\wedge dz_n)\wedge(d\overline{z}_2\wedge\cdots\wedge d\overline{z}_n)\\
&=-\frac{1}{2\pi}\mathrm{Im}\int_{|z_1|=\epsilon/h_i} f\frac{dz_1}{z_1}\wedge(dz_2\wedge\cdots\wedge dz_n)\wedge(d\overline{z}_2\wedge\cdots\wedge d\overline{z}_n)\\
&=-\frac{1}{2\pi}\mathrm{Im}\int_{|z_1|=\epsilon/h_i}\left(\int_{|z_i|<1\ ;\ i>1} f(0,z_{i>1})\wedge(dz_2\wedge\cdots\wedge dz_n)\wedge(d\overline{z}_2\wedge\cdots\wedge d\overline{z}_n)\right)\frac{dz_1}{z_1}\\
&=-\frac{1}{2\pi}\mathrm{Im}(2\sqrt{-1}\pi)\int_{z_1=0}\omega = \int_{D\cap U}\omega\\
&=-\int_{z_1=0}\omega = -\int_{D\cap U}\omega
\end{aligned}$$

$\square$