A Proof of a Theorem due to Erdös

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A *multiplicative function* is a function $f : \mathbb{N} \to \mathbb{R}$ such that for every $m, n \in \mathbb{N}$ such that

$$f(mn) = f(m)f(n) \text{ when } \gcd(m, n) = 1.$$

Let S be a multiplicative set, we say $f : S \to \mathbb{R}$ is quasi-multiplicative if for every $f(n^a) = f(n)^a$ for every $n \in S$ and $a \in \mathbb{N}$. In [1], Erdös proved the following theorem

Theorem 0.1. *If f is a multiplicative non-decreasing function, then there is $c \in \mathbb{R}_{>0}$ such that $f(n) = n^c$ for all $n \in \mathbb{N}$.*

The goal of this short note is to provide a more elementary proof. After proving the theorem, I found essentially the same proof, but technically simpler, in the article [2] by Everett Howe.

For the proof, we need two basic lemmas.

Lemma 0.2. *Let S be a multiplicatively closed subset of $\mathbb{N}$. Suppose that $f : S \to \mathbb{R}$ is quasi-multiplicative and non-decreasing. Then there is $c \in \mathbb{R}_{>0}$ such that $f(n) = n^c$ for all $n \in S$.*

Proof. Take $m, l \in S$ and write $f(m) = m^\beta$ and $f(l) = l^\gamma$. We want to prove that $\beta = \gamma$. To do this we choose $p_n, q_n \in \mathbb{N}$ such that $p_n \le q_n$ approaches $\frac{\log(m)}{\log(l)}$ from below as $n \to \infty$, that's

$$\frac{p_n}{q_n} \le \frac{\log(m)}{\log(l)}\ ,\ \lim_{n\to\infty} \frac{p_n}{q_n} = \frac{\log(m)}{\log(l)}.$$

Then for each n

$$p_n \log(l) \le q_n \log m \implies l^{p_n} \le m^{q_n} \implies f(l^{p_n}) \le f(m^{q_n}) \implies l^{\gamma p_n} \le m^{\beta q_n} \implies \frac{p_n}{q_n}\gamma \le \frac{\log m}{\log n}\beta,$$

and taking the limit as $n \to \infty$, we get $\gamma \le \beta$. We can prove in the same way that $\gamma \ge \beta$ by requiring p_n/q_n to converge to $\frac{\log(m)}{\log(l)}$ from above. □

Lemma 0.3. *Let f be a multiplicative function and non-decreasing and fix an odd prime p, then*

$$\liminf_{\substack{n\to\infty \\ n \text{ odd},\ n\equiv 1 \pmod p}} \frac{f(n+1)}{f(n)} = 1,$$

$$\liminf_{\substack{n\to\infty \\ n \text{ odd},\ n\equiv -1 \pmod p}} \frac{f(n+1)}{f(n)} = 1$$

Proof. We prove the first equality. Assume the result is not true. Since f is non-decreasing, then there is a constant $\epsilon > 0$ such that

$$\liminf_{\substack{n\to\infty \\ n \text{ odd},\ n\equiv 1 \pmod p}} \frac{f(n+1)}{f(n)} \ge 1 + 2\epsilon$$

It follows that there is a large $N > 0$ such that

$$\frac{f(n+1)}{f(n)} \ge 1 + \epsilon \tag{1}$$

whenever $n \geq N$ with n odd and $n \equiv 1 \pmod{p}$. Note that

$$f(2) = \frac{f(2p^k)}{f(p^k)} = \frac{f(2p^k)}{f(2p^k-1)} \cdot \frac{f(2p^k-1)}{f(2p^k-2)} \cdots \frac{f(p^k+1)}{f(p^k)}$$
$$= \prod_{j=0}^{p^{k-1}-1} \prod_{i=0}^{p-1} \frac{f(p^k+jp+i+1)}{f(p^k+jp+i)}.$$

Inequality (1) implies that for any $k \geq \log(N-1)/\log(p)$ and any $0 \leq j \leq p^{k-1}-1$, we have

$$\frac{f(p^k+jp+i+1)}{f(p^k+jp+i)} \geq \begin{cases} 1+\epsilon & \text{for } i=1 \\ 1 & \text{otherwise} \end{cases}.$$

Hence

$$f(2) \geq (1+\epsilon)^{p^{k-1}}$$

for all large k, which is a contradiction. This proves the first equation in the lemma, the second one can be proved in the same way. ☐

Corollary 0.4. *If f is multiplicative and non-decreasing, then $f(p^k) = f(p)^k$ for any odd prime p.*

Proof. We have the following

1. Let M be a positive integer such that $\gcd(M, p) = \gcd(M+1, p) = 1$, then

$$\begin{aligned} f(p^k)f(M) = f(p^kM) &\leq f(p^kM + p^k - p) \\ &= f(p(p^{k-1}M + p^{k-1} - 1)) \\ &\leq f(p)f(p^{k-1}M + p^{k-1}) \\ &= f(p)f(p^{k-1})f(M+1) \end{aligned}$$

   By taking $M = M_l = 2lp+1$ (we can because $\gcd(M+1, p) = \gcd(2kp+2, p) = \gcd(lp+1, p) = 1$), we get

$$f(p^k) \leq f(p)f(p^{k-1}) \cdot \frac{f(M_l+1)}{f(M_l)}.$$

   But $\{M_l\}$ is exactly the set of all numbers n such that n is odd and $n \equiv 1 \pmod{p}$. Now, taking $\liminf$ of both sides, the lemma implies

$$f(p^k) \leq f(p)f(p^{k-1}). \tag{2}$$

2. Let N be a positive integer such that $\gcd(N, p) = \gcd(N-1, p) = 1$, then

$$\begin{aligned} f(p^k)f(N) = f(p^kN) &\geq f(p^kN - p^k + p) \\ &= f(p(p^{k-1}N - p^{k-1} + 1)) \\ &\geq f(p)f(p^{k-1}N - p^{k-1}) \\ &= f(p)f(p^{k-1})f(N-1) \end{aligned}$$

   In the same way as above, by taking $N = N_l = 2lp-1$ as l varies and using the lemma, we get at the end

$$f(p)f(p^{k-1}) \leq f(p^k). \tag{3}$$

Now the corollary follows immediately by combining (2) and (3). ☐

Corollary 0.5. *Let S_{odd} be the multiplicative subset of $\mathbb{N}$ consiting of all odd numbers. If f is multiplicative and non-decreasing, then $f|_{S_{odd}}$ is quasi-multiplicative*

Proof. Let $n = p_1^{e_1} \cdots p_k^{e_k}$ be odd number written as product of coprime prime powers. Let a be a positive integer. Then by multiplicativity and corollary above we get

$$\begin{aligned} f(n^a) &= f(p_1^{ae_1} \cdots p_k^{ae_k}) \\ &= f(p_1^{ae_1}) \cdots f(p_k^{ae_k}) \\ &= f(p_1)^{ae_1} \cdots f(p_k)^{ae_k} \\ &= f(p_1^{e_1})^a \cdots f(p_k^{e_k})^a \\ &= f(n)^a \end{aligned}$$

□

Finally, we give a proof of the main theorem

Proof of Theorem 0.1. By the previous corollary and lemma 0.2, we find $c > 0$ such that $f(n) = n^c$ for all odd integers n. By monotonicity, we have the inequalities

$$f(2n-1)f(2^{k-1}) \leq f(2^k n) \leq f(2^k n + 2^{k-1}) = f(2^{k-1})f(2n+1)$$

By montonicity and multiplicativity we have for all odd n

$$f(n-2)f(2)f(2^{k-1}) = f(2n-4)f(2^{k-1}) \leq f(2^k)f(n) \leq f(2^{k-1})f(2n+4) = f(2^{k-1})f(2)f(n+2)$$

Substituting the form of f on odd integers and dividing by $f(n)$

$$\left(\frac{n-2}{n}\right)^c \cdot f(2)f(2^{k-1}) \leq f(2^k) \leq f(2)f(2^{k-1}) \cdot \left(\frac{n+2}{n}\right)^c$$

Now take $n \to \infty$ to obtain $f(2^k) = f(2)f(2^{k-1})$ and subsequently $f(2^k) = f(2)^k$. Now, it is immediate that f is quasi-multiplicative and as a result $f(n) = n^c$ for all positive integers n. □

References

[1] Paul Erdös, *On the Distribution Function of Additive Functions.* Annals of Mathematics (Second Series), 47(1) (1946) 1–20.

[2] Everett Howe, *A New Proof of Erdös's Theorem on Monotone Multiplicative Functions.* The American Mathematical Monthly, 93(8) (1986) 593–595.